Efficient Dead-Zone Plus Uniform Threshold Scalar Quantization of Generalized Gaussian Random Variables

Yizhou Duan, *Jun Sun, Jiaying Liu, Zongming Guo
Institute of Computer Science and Technology
Peking University
Beijing, China
{duanyizhou, sunjun, liujiaying, guozongming}@icst.pku.edu.cn

***Abstract*—This paper studies the rate-distortion (R-D) performance of entropy-constrained dead-zone plus uniform threshold scalar quantization (DZ+UTSQ) and nearly-uniform reconstruction quantization (NURQ) for generalized Gaussian distribution (GGD). We first derive the preliminary constraint of R-D optimized DZ+UTSQ/NURQ for GGD. Then for GGD source of actual DCT coefficients, the refined constraint and precise conditions of optimum DZ+UTSQ/NURQ are rigorously deduced in the real coding bit rate range. Based on above analysis, efficient DZ+UTSQ/NURQ design criteria are proposed to reasonably simplify the implementation of effective quantizer in practice.**

I. INTRODUCTION

In rate-distortion (R-D) analysis, considerable attentions have been focused on finding the best source distribution and designing the optimum entropy-constrained scalar quantizers. In the past decades, generalized Gaussian distribution (GGD) has been proposed as an excellent choice to represent actual DCT coefficients [1], [2]. And the Dead-zone plus uniform threshold quantization with nearly-uniform reconstruction quantization (DZ+UTSQ/NURQ) is now implemented by the major international video coding standards as recommendation [3]. The performance of DZ+UTSQ/NURQ is affected by two important parameter z and p, as is shown in Fig.1. The dead-zone ratio z is half size of the region with quantization index 0 to the quantization step s, and p is the reconstruction offset from the corresponding quantization index.

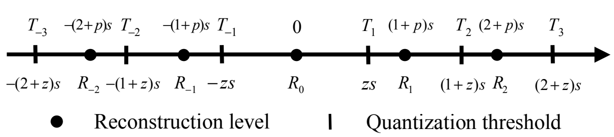

Figure 1. Illustration of the complete DZ+UTSQ/NURQ scheme

Although the integration of GGD and DZ+UTSQ/NURQ can accurately reflect the practical video coding system, to the best of our knowledge, there is no systematic study focusing on z and p of efficient DZ+UTSQ/NURQ for GGD. The only related work was provided by Gary [3], where the relationship of z and s was analyzed for Laplacian and Gaussian sources under theoretical optimal conditions. In addition, the performance comparison between uniform reconstruction quantization (URQ) and NURQ for DZ+UTSQ was provided and further extended to GGD. However, these contributions are not enough for real video applications due to the following reasons. First, there are still no available conclusions of z to guide the design of efficient DZ+UTSQ/NURQ for GGD. Second, the existing results of z and p were mostly derived at very high bit rate. The restrictive condition of real coding bit rate (below 1.0 bits/sample) lead to totally different selection of z and p, which well explains why Gary's proposal of DZ+UTSQ/URQ rounding technique to H.264/AVC only effectively improves coding efficiency in very high bit rate [3].

In this paper, we provide a thorough investigation into the R-D performance of DZ+UTSQ/NURQ for GGD. First, the preliminary constraint on z and p of optimum quantizer is derived for GGD. Then for the specific GGD source of actual DCT coefficients, z and p are further proved linear correlate in real coding bit rate. With all above prerequisites, the exact ranges of z and p required by optimum DZ+UTSQ/NURQ are obtained. And the criteria of effective DZ+UTSQ/NURQ design for practical video applications are also presented by fixing p and adjusting z within the proposed range.

The rest of the paper is organized as follows. In Section 2, DZ+UTSQ/NURQ is investigated with the probability density function (PDF) of GGD to deduce the preliminary constraint of z and p. In Section 3 and Section 4, the linear relationship and the optimal ranges of z and p are respectively derived. The efficient design criteria of DZ+UTSQ/NURQ for application are proposed in Section 5 and conclusions are drawn in Section 6.

II. PRELIMINARY CONSTRIANT OF Z AND P

The complete DZ+UTSQ/NURQ scheme is illustrated in Fig .1, which consists of the DZ+UTSQ *classification* rule and NURQ *reconstruction* rule [3], both are symmetric about zero. In classification, each real-valued input is selected by the quantization index according to *quantization threshold T*. In reconstruction, a real-valued *reconstruction level R* is produced for each index. For the quantization index $k \geq 1$, the decision interval is $[T_k, T_{k+1})$ and for $k \leq -1$, the interval is

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$(T_{-k-1}, T_{-k}]$. For the index 0, the interval is the dead-zone region given by (T_{-1}, T_1) . The quantization threshold T_k and the reconstruction level R_k are expressed as:

$$T_k = \begin{cases} (k-1+z)s, & k \geq 1 \\ T_{-k}, & k \leq -1 \end{cases} \quad \text{and} \quad R_k = \begin{cases} 0, & k = 0 \\ (k+p)s, & k \geq 1 \\ R_{-k}, & k \leq -1 \end{cases} . \quad (1)$$

For the symmetric quantization scheme, we first consider only *positive* axis for convenience, without loss of generality to the similar problem posed for the negative axis.

Throughout our work, we refer optimum quantizer as the one minimizes the average distortion under given limit of output entropy rate. The necessary conditions for optimum quantizer were formulated in [4], represented by the well-known conclusion from Max [5] under the mean square error (MSE) measure: for any distribution, the reconstruction level R_k should locate at the focus of the PDF $p(x)$ in interval $[T_k, T_{k+1})$. Thus, the optimum DZ+UTSQ/NURQ requires that

$$R_k = \int_{(k+z-1)s}^{(k+z)s} xp(x)dx \Big/ \int_{(k+z-1)s}^{(k+z)s} p(x)dx, \; k \geq 1 . \quad (2)$$

Since $x > 0$ and $p(x) \geq 0$, it is obvious that

$$\begin{aligned} R_k &> (k+z-1)s \cdot \int_{(k+z-1)s}^{(k+z)s} p(x)dx \Big/ \int_{(k+z-1)s}^{(k+z)s} p(x)dx \\ &= (k+z-1)s \end{aligned} , \quad (3)$$

and

$$\begin{aligned} R_k &< (k+z)s \cdot \int_{(k+z-1)s}^{(k+z)s} p(x)dx \Big/ \int_{(k+z-1)s}^{(k+z)s} p(x)dx \\ &= (k+z)s \end{aligned} . \quad (4)$$

From (1), (3) and (4) the result of $z < 1 + p < z + 1$ can be directly obtained. Specifically in some regions where $p(x) = 0$ everywhere, R_k should locate at $(k+z-1)s$. Thus we have

$$z \leq 1 + p < z + 1 . \quad (5)$$

The expression (5) is the constraint on z and p of optimum DZ+UTSQ/NURQ for all types of distributions. To refine (5) for GGD, we first prove the important inequality of

$$\int_{(k+z-1)s}^{(k+z)s} xp(x)dx \leq (k+z-\frac{1}{2})s \cdot \int_{(k+z-1)s}^{(k+z)s} p(x)dx, \; k \geq 1 . \quad (6)$$

In arbitrary positive interval $[a, b)$ we have

$$\begin{aligned} \int_a^b xp(x)dx &= \int_a^{(a+b)/2} xp(x)dx + \int_{(a+b)/2}^b xp(x)dx \\ &= \int_0^{(b-a)/2} [(x+a)p(x+a) + (b-x)p(b-x)]dx \end{aligned} . \quad (7)$$

The PDF of GGD is written as

$$p(x) = \frac{\alpha \eta(\alpha, \beta)}{2\Gamma(1/\alpha)} \exp\left(-\left[\eta(\alpha, \beta)|x|\right]^{\alpha}\right) , \quad (8)$$

with

$$\eta(\alpha, \beta) = \beta^{-1}\left(\Gamma(3/\alpha) / \Gamma(1/\alpha)\right)^{1/2} \text{ and } \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} d_t , \quad (9)$$

where α is the shape parameter and β is the root variance. The PDF of GGD is monotonically decreasing in positive axis, indicating $[(x+a)-(b-x)][p(x+a)-p(b-x)] \leq 0$ when $x \in [0, (b-a)/2]$, by which it is easy to deduce that

$$(x+a)p(x+a) + (b-x)p(b-x) \leq \frac{(a+b)[p(x+a)+p(b-x)]}{2} . \quad (10)$$

Using (7) and (10), the inequality of (6) is easily proved. Then with (1) and (3), it is deduced that

$$\begin{aligned} R_k &\leq (k+z-\frac{1}{2})s \cdot \int_{(k+z-1)s}^{(k+z)s} p(x)dx \Big/ \int_{(k+z-1)s}^{(k+z)s} p(x)dx \\ &= (k+z-\frac{1}{2})s \end{aligned} . \quad (11)$$

Finally, based on (1) and (11), the general constraint of (5) is refined into

$$z \leq 1 + p \leq z + \frac{1}{2} , \quad (12)$$

which specifies the preliminary constraint on z and p of optimum DZ+UTSQ/NURQ for GGD. In next section, on basis of (12), the stricter constraint of z and p are further derived for optimum quantizer design.

III. LINEAR CONSTRAINT OF Z AND P

For optimum quantizers, reconstruction level is required to locate at the focus of the PDF in each interval, implying that different selections of z and p can result in similar R-D performance. It is verified that most actual DCT coefficients have GGD shape α between 0.5 and 1.0 [2], and the GGD with α between 0.1 and 1.0 provides a useful model of broad-tailed process [4]. Thus for GGD of real sources, in intervals other than the dead-zone region, the PDF can be regarded linear with a slope approximating 0 which is determined by α . Taking interval $[T_k, T_{k+1})$ for example, for an optimum quantizer, R_k is the focus that divides the region into two parts $\overline{T_k R_k}$ and $\overline{R_k T_{k+1}}$, as is shown in Fig. 2. With the given α , the slope of GGD PDF in $[T_k, T_{k+1})$ can be treated as fixed value, thus, $\overline{T_k R_k} / \overline{R_k T_{k+1}}$ should keep constant. Using z_1 , p_1 and z_2 , p_2 to represent two groups of different z and p, we have

$$\frac{(k+p_1)-(k-1+z_1)}{(k+z_1)-(k+p_1)} = \frac{(k+p_2)-(k-1+z_2)}{(k+z_2)-(k+p_2)} , \quad (13)$$

From (13) we can derive that $p_1 - p_2 = z_1 - z_2$, which means z and p are linear correlate. Based on (12), the stricter linear constraint on z and p is deduced

$$z - p = c, \;\; 1/2 \leq c \leq 1 , \quad (14)$$

where c is constant.

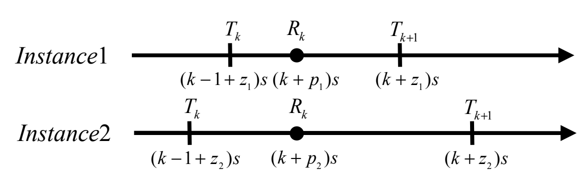
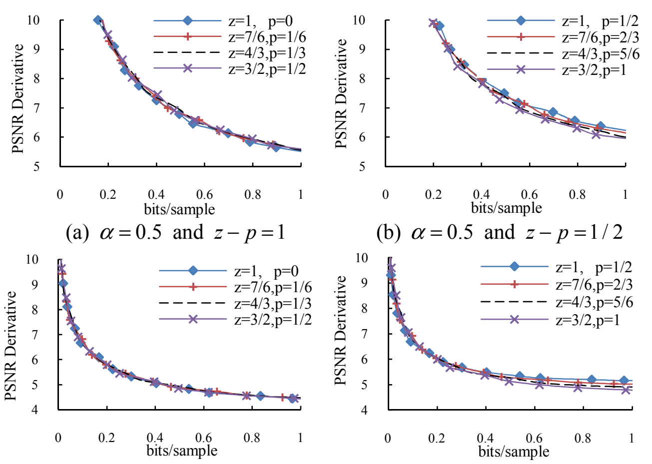


Figure 2. Illustration of two groups of z and p that result in different DZ+UTSQ/NURQ instances with similar (optimal) R-D performance

Simulations of the distortion-rate (D-R) derivative curves with GGD shapes 0.5 and 1.0 under the linear constraint are observed through four typical patterns $z - p = 1/2$, $z - p = 2/3$, $z - p = 5/6$ and $z - p = 1$. Related results for $z - p = 1/2$ and $z - p = 1$ in the real coding bit rate range are shown in Fig. 3, which shows that DZ+UTSQ/NURQ conforming to the same pattern are of similar performance. More specifically, the curves generated by $z - p = 1$ coincide perfectly. And for the instances satisfying $z - p = c, 1/2 \le c < 1$, the R-D difference accumulates as bit rate increases, from which we can conclude that under the same pattern of $z - p = c, 1/2 \le c < 1$, the quantizer with smaller z provides better R-D efficiency.

To sum up, the linear constraint of (14) is a good approximation of the relationship between z and p in the real coding bit rate range for GGD. Different DZ+UTSQ/NURQ designs conforming to the same pattern of z and p lead to similar efficiency, thus the optimum DZ+UTSQ/NURQ design for GGD can be simplified to the selection of the DZ+UTSQ/NURQ patterns with optimal R-D performance. Since z and p are linear correlate, for convenience it is reasonable to represent each DZ+UTSQ/NURQ pattern by fixing one of z and p and properly adjusting the other one. Therefore, the R-D comparison between different patterns can be further simplified to the observation of the R-D performance under DZ+UTSQ/NURQ where only one of z and p is adjusted. In this case, there are still two questions to be answered: 1), what are the ranges of z and p in which they can be fixed or adjusted; 2), how to decide which one of z and p should be fixed and which one should be adjusted. In the following sections we give answers to the above questions. For the first one, the optimal ranges of z and p are rigorously deduced, and for the second one, the strategy is presented by fixing p and adjusting z within the proposed range.



(a) $\alpha = 0.5$ and $z - p = 1$ (b) $\alpha = 0.5$ and $z - p = 1/2$

(c) $\alpha = 1.0$ and $z - p = 1$ (d) $\alpha = 1.0$ and $z - p = 1/2$

Figure 3. D-R derivative curve of GGD under the linear constraint of optimum DZ+UTSQ/NURQ in real coding bit rate range

IV. OPTIMAL RANGES OF Z AND P

Since the linear constraint still provides infinite ways to select z and p, it is meaningful to further excavate the exact ranges of z and p for optimum DZ+UTSQ/NURQ with GGD. In [6], for Laplacian distribution (GGD shape 1.0), Gary derived the range of z in the design of optimum UTSQ, known as $1/2 < z < 1$. Here we rigorously deduce the comprehensive conclusion of z and p for optimum DZ+UTSQ/NURQ with the GGD source of actual DCT coefficients in the real coding bit rate range.

First, with the linear constraint (14) and $p \ge 0$, it is easy to know $z \ge 1/2$. When $z = 1/2$, the DZ+UTSQ degrades into the special case of UTSQ, a well-known quantization scheme. In the earlier works [6]–[9], UTSQ was proved to be optimum at very high bit rate, while in relatively lower bit rate the requirement of infinite thresholds cannot be satisfied. Hence in real coding bit rate, UTSQ is not the choice for optimality, which refines the range of z into $z > 1/2$.

On the other hand, the NURQ degrades into the special case of URQ by fixing p to 0. More particularly, when $z = 1$ and $p = 0$ are both satisfied, URQ further degrades into uniform reconstruction with unity ratio quantizer (URURQ). URQ and URURQ are both widely used for their simplicity in design and operation. It has been verified by Gary that URURQ gives the lower bound on the performance of the best URQ [6], while URQ is proved sub-optimal for GGD [3]. Although for GGD, the R-D difference between URQ and optimum NURQ is negligible [3], to be strict, URQ is still not the choice for real optimality, not to mention URURQ. Note that URURQ conforms to $z - p = 1$. Since the quantizers conforming to the pattern $z - p = 1$ with the same α are of exactly the same R-D performance, it is reasonable to use $z - p = 1$ as a benchmark of sub-optimal efficiency to evaluate other DZ+UTSQ/NURQ patterns in real coding bit rate. As a result, $z - p = 1$ and $p = 0$ are both excluded from the optimal ranges, which makes the interim result of $p > 0$ and refines the linear constraint into $z - p = c, 1/2 \le c < 1$.

Furthermore, the conclusion of the linear constraint shows that under the same pattern of $z - p = c, 1/2 \le c < 1$, the better R-D performance is obtained by the quantizer with smaller z. Thus for any $z > 1$ we can find a new value of $z' \le 1$ and the corresponding p' satisfying $z' - p' = c$ with better R-D efficiency. Therefore, the optimal performance is achieved only when $z \le 1$, which also implicates $p \le 1/2$.

Through above rigorous deduction, we finally derive the optimal ranges of z and p for DZ+UTSQ/NURQ with GGD:

$$\frac{1}{2} < z \le 1, \tag{15}$$

and

$$0 < p \le \frac{1}{2}, \tag{16}$$

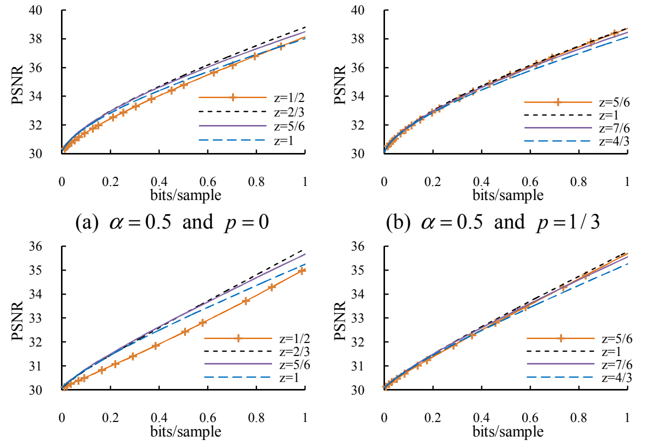
both are conforming to the linear constraint of (14).

V. EFFICIENT DZ+UTSQ/NURQ DESIGN FOR PRACTICE

The optimum ranges have provided an acceptable scope to decide the values of z and p. However, in video coding application, different standards support different proposals of setting p for quantization [3], which still causes confusion. To inspire a rational solution of efficient DZ+UTSQ/NURQ design for practice, we carefully analyze the D-R curves of GGD in the real coding bit rate range through two groups of simulation experiments.

The first group of experiments is designed to evaluate the influence of z on R-D performance. For both GGD shapes 0.5 and 1.0, four p values 0, 1/6, 1/3 and 1/2 are selected, each generating four D-R curves from four patterns: $z - p = 1/2$, $z - p = 2/3$, $z - p = 5/6$ and $z - p = 1$. In this way, z is the only variable to affect performance. Related results for $p = 0$

and $p=1/3$ are illustrated in Fig. 5, and the conclusions referred in our previous analysis are also verified. First, the UTSQ represented by $z=1/2$ is shown in Fig. 4(a) and 4(c), which is of explicitly poor performance in real coding bit rate range. Second, Fig. 4 illustrates the R-D comparison between $z-p=1$ and other patterns, which confirms that optimum DZ+UTSQ/NURQ should at least outperform the benchmark of URURQ. The first group of experiments well demonstrates that R-D performance is greatly affected by the selection of z. The superiority in R-D performance is achieved only when z approaches the optimal range of (15).



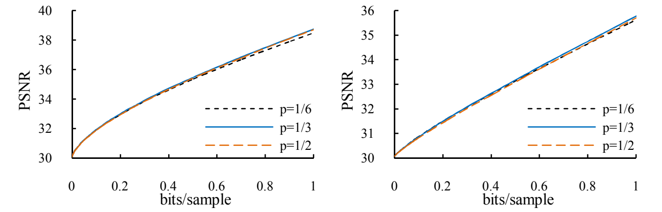
(a) $\alpha=0.5$ and $p=0$ (b) $\alpha=0.5$ and $p=1/3$

(c) $\alpha=1.0$ and $p=0$ (d) $\alpha=1.0$ and $p=1/3$

Figure 4. D-R curves of GGD under four different DZ+UTSQ/NURQ patterns in real coding bit rate range

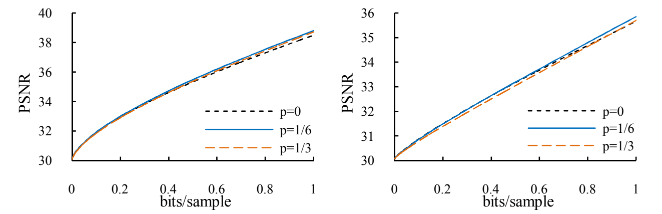
The second group of experiments is designed to evaluate the influence of p on the optimal R-D performance, which is further divided into two parts. The former part is to observe the R-D difference of various p within the optimal range of (16), while the latter one is for the evaluation of URQ ($p=0$). In each part, three D-R curves generated by three patterns $z-p=1/2$, $z-p=2/3$, $z-p=5/6$ are exhibited for both GGD shapes 0.5 and 1.0. To keep optimality, z is required to be within the range of (15). For convenience, we fix z to 1 in the former part and 5/6 in the latter one. Corresponding results are illustrated in Fig. 5 and Fig. 6 respectively. It is explicit in Fig. 5 that for the same z, different p satisfying (16) result in very similar R-D performances. In addition, Fig. 6 shows that the R-D difference between optimum DZ+UTSQ/NURQ and URQ is ignorable. The second group of experiments implies that when z is within the optimal range, different p from $0\le p\le 1/2$ will not affect the optimality of coding efficiency.

In the end, the efficient DZ+UTSQ/NURQ design criteria for practice are completed. Normally, we can fix p to an arbitrary value satisfying the optimal range $0<p\le 1/2$ and merely adjust z within the range $1/2<z\le 1$ to achieve optimal R-D performance. For simplicity, we can also choose URQ by simply setting p to 0. In this case, we need to keep z in the range of $1/2<z<1$ to avoid URURQ and obtain nearly-optimal coding efficiency. The convenience and effectiveness of URQ are the most important reason of its prevalence in the latest video coding standard H.264/AVC. In sum, with our proposed solution, the practical implementation of efficient DZ+UTSQ/NURQ is effectively and reasonably simplified.



(a) $\alpha=0.5$ and $z=1$ (b) $\alpha=1.0$ and $z=1$

Figure 5. D-R curves of GGD under three different patterns of optimum DZ+UTSQ/NURQ in real coding bit rate range



(a) $\alpha=0.5$ and $z=5/6$ (b) $\alpha=1.0$ and $z=5/6$

Figure 6. D-R curves of GGD under three different patterns of efficient DZ+UTSQ/NURQ in real coding bit rate range

VI. Conclusion

In this paper, we provide a detailed performance analysis of R-D optimized DZ+UTSQ/NURQ for GGD. There are two main points we have contributed in this work. First, we rigorously deduce the optimal ranges of z and p for GGD of actual DCT coefficients in real coding bit rate range. Second, we propose the efficient DZ+UTSQ/NURQ design criteria which reasonably simplify the implementation of effective quantizer in practice. All the above results bring us insights into the high-performance DZ+UTSQ/NURQ scheme with GGD random variables.

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